

Uncertainty Quantification in Fatigue Crack Growth Prognosis

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ABSTRACT

This paper presents a methodology to quantify the uncertainty in fatigue crack growth prognosis, applied to structures with complicated geometry and subjected to variable amplitude multi-axial loading. Finite element analysis is used to address the complicated geometry and calculate the stress intensity factors. Multi-modal stress intensity factors due to multi-axial loading are combined to calculate an equivalent stress intensity factor using a characteristic plane approach. Crack growth under variable amplitude loading is modeled using a modified Paris law that includes retardation effects. During cycle-by-cycle integration of the crack growth law, a Gaussian process surrogate model is used to replace the expensive finite element analysis. The effect of different types of uncertainty – physical variability, data uncertainty and modeling errors – on crack growth prediction is investigated. The various sources of uncertainty include, but not limited to, variability in loading conditions, material parameters, experimental data, model uncertainty, etc. Three different types of modeling errors – crack growth model error, discretization error and surrogate model error – are included in analysis. The different types of uncertainty are incorporated into the crack growth prediction methodology to predict the probability distribution of crack size as a function of number of load cycles. The proposed method is illustrated using an application problem, surface cracking in a cylindrical structure

1. INTRODUCTION

The scientific community has increasingly resorted to the use of computational models to predict the performance of

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engineering components and systems so as to facilitate risk assessment and management, inspection and maintenance scheduling, and operational decision-making. Model-based prognosis, i.e. predicting the performance of a system using a physics-based model is promising for health management. However, no model can perfectly represent the system and hence it is necessary to include model form errors and model uncertainty in the prognosis. Secondly, complex engineering systems may have to be modeled using multiple models that interact with one another. In such cases, each model has its own sources of error/uncertainty and the interaction between the errors of multiple models is non-trivial. Some errors are deterministic while some others are stochastic. Systematic methods are needed to quantify the uncertainty and confidence associated with the model prediction. Hence, prognosis methods need the following capabilities: (1) integration of multiple models, (2) quantification of different types of uncertainty and error (physical variability, data uncertainty, and model uncertainty), and (3) integration of the various types of uncertainty to calculate the overall uncertainty in the results of prognosis.

This paper develops an uncertainty quantification methodology to meet the above needs, and uses the problem of fatigue crack growth to illustrate such development. The objective of this problem is to predict the crack growth in a structural component as a function of number of load cycles. Mechanical components in engineering systems are often subjected to cyclic loads leading to fatigue, crack initiation and progressive crack growth. It is essential to predict the performance of such components to facilitate risk assessment and management, inspection and maintenance scheduling and operational decision-making. Researchers have pursued two different kinds of methodologies for fatigue life prediction. The first method is based on material testing (to generate S-N, ϵ -N curves) and use of an

assumed damage accumulation rule. In this method, specimens are subjected to repeated cyclic loads under laboratory conditions. Hence the results are specific to the geometry of the structure as well as the nature of loading. Further, the performance of these components under field conditions is significantly different from laboratory observation, due to various sources of uncertainty accumulating in the field that render experimental studies less useful. Hence, this methodology cannot be used directly to predict the fatigue life of practical applications wherein complicated structures subjected to multi-axial loading.

The second method for fatigue life prediction is based on principles of fracture mechanics and crack growth analysis. A crack growth law is assumed and the progressive growth of the crack is modeled. However, this is not straightforward. Fatigue crack growth is a stochastic process and there are different types of uncertainty – physical variability, data uncertainty and modeling errors, associated with it. Uncertainty appears at different stages of analysis and the interaction between these sources of uncertainty cannot be modeled easily. Further, the application of crack growth principles to complicated structures, subjected to multi-axial variable amplitude loading requires repeated evaluation of finite element analysis which makes the computation expensive.

Some of these problems have been investigated by researchers in detail. The first problem in using a crack growth model is that the initial crack size is not known. This issue is further complicated by the fact that small crack growth propagation is anomalous in nature. This problem was addressed by the introduction of an equivalent initial flaw size (EIFS) nearly thirty years ago. The concept of EIFS was introduced to by-pass small crack growth analysis and to substitute an initial crack size in long crack growth models such as Paris' law. However EIFS does not represent any physical quantity and cannot be measured using experiments. Initially, certain researchers used empirical crack lengths between 0.25 mm and 1 mm for metals (JSSG, 1998; Gallagher et al., 1984; Merati et al., 2007). Later, several researchers (Yang, 1980; Moreira et al., 2000; Fawaz, 2000; White et al., 2005; Molent et al., 2006) used back-extrapolation techniques to estimate the value for equivalent initial flaw size. Recently, Liu and Mahadevan (2008) proposed a methodology based on the Kitagawa-Takahashi diagram (Kitagawa and Takahashi, 1976) and the El-Haddad Model (Haddad et al., 1979) to derive an analytical expression for the equivalent initial flaw size. The current research work uses this concept to calculate the statistics of EIFS from material properties such as threshold stress intensity factor and fatigue limit. These material properties are calculated from experimental data and the associated data uncertainty due to measurement errors, sparseness of data, etc. needs to be taken into account.

The next step in fatigue crack growth prognosis is to choose a crack growth model. There are many crack growth models available in literature. In this paper, a modified Paris law is used as the crack growth law for the sake of illustration, but an error term (treated as a random variable) is added to represent the fitting error since experimental data were used to estimate the coefficients of the Paris model. Further, the model coefficients are also treated as random variables. The effects of variable amplitude loading are considered by including retardation effects along with modified Paris' law. Several models (Wheeler, 1972; Schjive, 1976; Noroozi et al., 2008) have been proposed to tackle variable amplitude loading conditions and this paper uses Wheeler's retardation model (Wheeler, 1972) only for illustration purposes. Further, only coplanar cracks have been considered for analysis.

The modified Paris law based on linear elastic fracture mechanics calculates the increase in crack size as a function of the stress intensity factor, during each loading cycle. The stress intensity factor, in turn, is a function of the current crack size, crack configuration, geometry of the structural component and loading conditions. If structures with complicated geometry are subjected to multi-axial loading, then the stress intensity factor needs to be calculated through expensive finite element analysis, at every loading cycle. This paper replaces the finite element analysis with a surrogate model, known as the Gaussian process (GP) interpolation. Several finite element analysis runs are used to train this surrogate model and then, the surrogate model is used to predict the stress intensity factor, to be used in the crack growth law. There are two types of errors in this procedure. First, the finite element analysis has discretization error that needs to be accounted for while training the surrogate model. Second, the surrogate model adds further uncertainty since it is obtained by fitting the model to the (finite element) training data.

In addition to the above mentioned model uncertainty and data uncertainty (used to calculate the EIFS), natural variability in many input variables introduces uncertainty in model output. The loading on the structure is usually random in nature. A variable amplitude multi-axial loading history consisting of bending and torsion is illustrated in this paper. Natural variability also includes variability in material properties, geometry and boundary conditions. The variability in certain material properties such as fatigue limit and threshold stress intensity factor is considered while deriving the statistical distribution of EIFS. The geometry of the specimen and boundary conditions are considered deterministic in this research work.

The main focus of this paper is to investigate in detail each source of uncertainty and propose a methodology that can effectively account for all of them. Finally, the developed framework is used to predict the probabilistic fatigue life of the structure.

The next section reviews the existing literature on this topic and motivates the current study. Section 3 presents the algorithm used in this paper to predict the fatigue life of structures with complicated geometry and subjected to variable amplitude, multi-axial loading. The various sources of uncertainty in this procedure are discussed in Section 4. Section 5 presents the proposed framework for uncertainty quantification in crack growth prediction. Section 6 illustrates the methodology through an example, considering cracking in a cylindrical structure.

2. LITERATURE REVIEW

Numerous studies have dealt with methods for uncertainty quantification in prognosis. Most of these studies have focused mainly on natural variability; sources of data uncertainty and model uncertainty have not been considered in detail. Hemez (2005) discusses uncertainty quantification in prognosis and uncertainty propagation techniques that deal only with physical variability. Chelidze and Cusumano (2004) demonstrate a dynamical systems approach to prognosis in an electromechanical system and estimate the uncertainty in the prognosis results. Saha and Goebel (2007) discuss diagnostics and prognostics of batteries using Bayesian techniques. Medjaher et al (2009) use dynamic Bayes networks for prognosis of industrial systems. These studies mostly consider physical variability; modeling errors and their sources are not analyzed in detail; and data uncertainty due to sparse data is not considered.

The “damage prognosis” project at Los Alamos national laboratory (Doebling and Hemez, 2001; Hemez et al., 2003; Farrar et al., 2004; Farrar and Lieven, 2006) considered the problem of fatigue cracking in detail and proposed sampling techniques to predict crack growth in composite plates; the error between prediction and observation was also characterized. Loading (uniaxial impact loading) conditions and geometric and material properties are treated as random variables. Surrogate models were used to replace expensive finite element models, and included in a sampling based framework for uncertainty propagation. Finite element analysis results were used to train the surrogate models, but the discretization error was not quantified. Further, the errors due to usage of surrogate models, errors in crack growth model, etc. were not addressed.

Besterfield et al. (1991) combined probabilistic finite element analysis with reliability analysis to predict crack growth in plates. Random mixed mode loading cycles, physical variability in material properties, randomness in crack configuration (size, position and angle) were considered. However, the implementation of probabilistic finite element analysis is computationally expensive for structures with complicated geometry. Other sources of uncertainty such as data uncertainty and model uncertainty were not considered.

Patrick et al (2007) introduced an online fault diagnosis and failure prognosis methodology applied to a helicopter transmission component. A crack growth model (Paris law) was used for fatigue life prediction. Bayesian techniques were implemented to infer the initial crack size, which was used for probabilistic fatigue life prediction using particle filter techniques. Other sources of uncertainty such as error in Paris law, variability in model parameters, and randomness in loading were not considered.

Gupta and Ray (2007) developed algorithms for online fatigue life estimation that relied on time series data analysis of ultrasonic signals and were built on the principles of symbolic dynamics, information theory and statistical pattern recognition. Physical variability in material geometry (surface defects, voids, inclusions, sub-surface defects), minor fluctuations in environmental conditions and operating conditions were used to quantify the uncertainty in detection which was further used to quantify the uncertainty in prognosis.

Pierce et al (2007) discussed the application of interval set techniques to the quantification of uncertainty in a neural network regression model of fatigue life, applied to glass fiber composite sandwich materials. This paper only considered the uncertainty in input data and other sources of uncertainty were not investigated in detail.

Orchard et al (2008) used the method of particle filters for uncertainty management in fatigue prediction. However, the various sources of uncertainty were not clearly delineated and considered in the analysis. While the use of conditional probability has been recommended for probabilistic predictions, this turns out to be expensive when variable amplitude loading cycles are considered, as the ensemble of predictions grows in size as a function of the number of loading cycles.

Papazian et al (2009) developed a structural integrity prognosis system (SIPS), based on collaboration between sensor systems and advanced reasoning methods for data fusion and signal interpretation, and modeling and simulation. Probabilistic principles such as likelihood and conditional probability were used to compare model predictions and sensor data. While measurement errors and sensor data were considered in detail, solution errors, variability of model parameters, randomness in loading, etc were not considered.

Thus, past studies on uncertainty quantification in prognosis have ignored several sources of uncertainty or not investigated them in detail. Physical variability (such as randomness in loading conditions, material properties, etc.) has been mainly studied by researchers, whereas other sources of uncertainty such as data uncertainty and model uncertainty have not been fully addressed.

This paper proposes a framework which can effectively account for different sources of uncertainty – physical variability, data uncertainty, and model uncertainty.

Data uncertainty arises due to the use of sparse data to construct probability distributions for input parameters. In previous studies, the input parameters are usually assumed to have completely known distributions (usually, normal) and prognosis is carried out using these distributions. In some cases, there may be enough data available to quantify such precise distributions. However, in many cases, it is impossible to construct precise probability distributions using a few available data (sparse data). Hence, there is uncertainty in the probability distributions constructed from such data. Therefore, this paper proposes a methodology where (1) the uncertainty in the distribution parameters is quantified using a resampling technique, and (2) an overall unconditional probability distribution of the quantity of interest, that includes the contribution of data uncertainty, is calculated.

When model-based methods are used for prognosis, it is essential to account for the different types of model uncertainty and errors. The significance of model uncertainty increases when there are multiple interacting models, because the quantification of the combined effect of the different sources of model uncertainty is non-trivial. In this paper, the algorithm for crack growth propagation uses multiple interacting models – finite element model, surrogate model, crack growth law, etc. First, the paper proposes methods to quantify the uncertainty/error in each of the individual models. It is important to note that some errors are deterministic (finite element discretization error) while some others are stochastic (crack growth law uncertainty, surrogate model prediction uncertainty), and they occur at different stages of the analysis. Therefore, the quantification of the overall uncertainty due to the combination of multiple sources of model uncertainty/errors is not trivial. This paper proposes a methodology for overall uncertainty quantification, where deterministic errors are addressed by correcting and stochastic errors are addressed through sampling.

Thus the contributions of this paper are: (1) connect different models such as finite element model, surrogate model, crack growth law, etc. efficiently; (2) quantify the uncertainty in each model separately; (3) treat deterministic model errors and stochastic model errors separately; (4) include physical variability as well as data uncertainty; and (5) quantify the overall uncertainty in crack growth prediction by correcting deterministic errors and sampling stochastic errors. The major advantage of the proposed methodology is that it provides a framework for including not only physical variability, but also data uncertainty and multiple model errors (both deterministic and stochastic). The various sources of uncertainty are discussed in detail,

later in Section 4. Prior to that, the algorithm for crack growth propagation is outlined in the following section.

3. CRACK GROWTH PROPAGATION

Consider the growth of an elliptic crack. A schematic of the crack growth is shown in Figure 1.

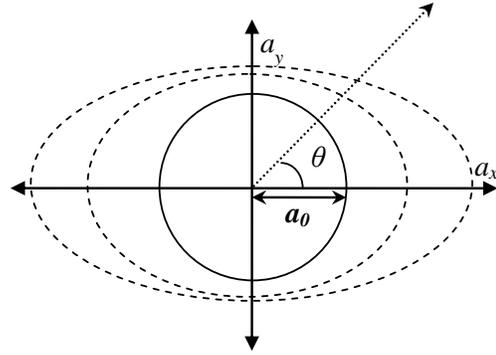


Figure 1. Elliptic Crack Growth

In Figure 1, a_x denotes the length of the semi-major axis and a_y denotes the length of the semi-minor axis. The aspect ratio, calculated as ratio between a_x and a_y is denoted by γ . If θ denote the angle of orientation, then a_x corresponds to $\theta = 0^\circ$ and a_y corresponds to $\theta = 90^\circ$. Crack growth laws such as Paris law (applicable to long cracks) predict the increase in crack size as a function of stress intensity factor, which in turn depends on the current crack size (a_x, a_y), aspect ratio (γ), angle of orientation (θ) and loading (L). In this paper, a has been used to denote the crack size in two directions, i.e. $\mathbf{a} = [a_x, a_y]$. Hence, the two dimensional array \mathbf{a} contains information about aspect ratio (β) as well. Starting with an initial crack size (\mathbf{a}_0), the growth of the crack can be modeled and the crack size after a given number of cycles can be calculated. However, the initial crack size cannot be calculated exactly. The concept of EIFS was proposed to tackle this problem. Starting with the introduction of EIFS, this section explains the various steps involved in using a crack growth model to predict the crack size as a function of number of cycles.

3.1 Use of EIFS in Crack Growth Law

The rigorous approach to fatigue life prediction would be to perform crack growth analysis starting from the actual initial flaw, accounting for voids and non-metallic inclusions. If the initial crack size is large, then long crack growth models such as Paris' law can be used directly. However, this is not the case in most materials. Hence the long crack growth model cannot be used directly. A schematic plot of the long crack and short crack growth curves is given in Figure 2.

This paper uses a long crack model for fatigue crack growth analysis; the short crack growth calculations are bypassed

through the use of an equivalent initial flaw size, as explained later in this section.

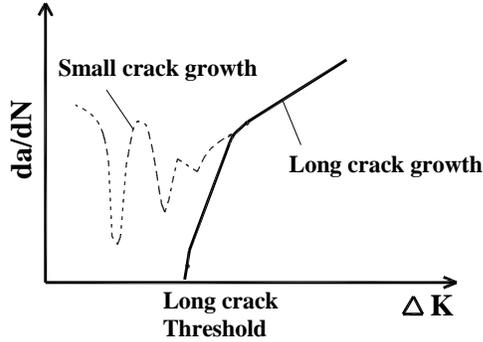


Figure 2. Schematic of Crack Growth

Consider any long crack growth law used to describe the relationship between da/dN and ΔK , where N represents the number of cycles, a represents the crack size and ΔK represents the stress intensity factor. This paper uses a modified Paris' law with Wheeler's retardation model as:

$$da/dN = \phi^r C (\Delta K)^n (1 - \Delta K_{th}/\Delta K)^m \quad (1)$$

Note that several models (Wheeler, 1972; Schjive, 1976; Noroozi et al., 2008) have been proposed to tackle variable amplitude loading conditions. This paper uses a Wheeler's retardation model (Wheeler, 1972) only to illustrate the proposed uncertainty quantification methodology, and other appropriate models can also be used instead of the Wheeler model. In Eq. (1), ϕ^r refers to the retardation parameter (Sheu et al., 1995), and is equal to unity if $a_i + r_{p,i} > a_{OL} + r_{p,OL}$ where a_{OL} is the crack length at which the overload is applied, a_i is the current crack length, $r_{p,OL}$ is the size of the plastic zone produced by the overload at a_{OL} , and $r_{p,i}$ is the size of the plastic zone produced at the current crack length a_i . Else, ϕ^r is calculated as shown in Eq. (2).

$$\phi^r = (r_{p,i} / (a_{OL} + r_{p,OL} - a_i))^\lambda \quad (2)$$

In Eq. (2), λ is the curve fitting parameter for the original Wheeler model termed the shaping exponent (Yuen et al., 2006). Song et al. (2001) observed that crack growth retardation actually takes place within an effective plastic zone. Hence the size of the plastic zone can be calculated in terms of the applied stress intensity factor (K) and yield strength (σ) as:

$$r_p = \alpha (K/\sigma)^2 \quad (3)$$

In Eq. (3), α is known as the effective plastic zone size constant which is calculated experimentally (Yuen et al., 2006). The retardation model parameters are calibrated for

particular experimental conditions, which need to be matched to the problem at hand for proper application. The expressions in Eq. (2) and Eq. (3) can be combined with Eq. (1) and used to calculate the crack growth as a function of number of cycles. In each cycle, the stress intensity factor can be expressed as a function of the crack size (a), loading (L) and angle of orientation (θ). Hence, the crack growth law in Eq. (1) can be rewritten as:

$$da/dN = g(a, L, \theta) \quad (4)$$

The concept of an equivalent initial flaw size was proposed to bypass small crack growth analysis and make direct use of a long crack growth law for fatigue life prediction. The equivalent initial flaw size, a_0 is calculated from material properties (ΔK_{th} , the threshold stress intensity factor and σ_f , the fatigue limit) and geometric properties (Y) as explained in Liu and Mahadevan (2008).

$$a_0 = (1/\pi)(\Delta K_{th}/Y\sigma_f)^2 \quad (5)$$

By integrating the expression in Eq. (1), the number of cycles (N) to reach a particular crack size a_N can be calculated as shown in Eq. (6).

$$N = \int dN = \int 1/(\phi^r C (\Delta K)^n (1 - \Delta K_{th}/\Delta K)^m) da \quad (6)$$

For structures with complicated geometry and loading conditions, the integral in Eq. (6) is to be evaluated cycle by cycle, calculating the stress intensity factor in each cycle of the crack growth analysis. The calculation of the stress intensity factor is explained in the following subsection.

3.2 Calculation of Stress Intensity Factor

The stress intensity factor ΔK in Eq. (6) can be expressed as a closed form function of the crack size for specimens with simple geometry subjected to constant amplitude loading. However, this is not the case in many mechanical components, where ΔK depends on the loading conditions, geometry and the crack size. Further, if the loading is multi-axial (for example, simultaneous tension, torsion and bending), then the stress intensity factors corresponding to three modes need to be taken into account. This can be accomplished using an equivalent stress intensity factor. If K_I , K_{II} , K_{III} represent the mode-I, mode-II and mode-III stress intensity factors respectively, then the equivalent stress intensity factor K_{eqv} can be calculated using a characteristic plane approach proposed by Liu and Mahadevan (2008). The use of the characteristic plane approach for crack growth prediction under multi-axial variable amplitude loading has been validated earlier with several data sets.

During each cycle of loading, the crack grows and hence, the stress intensity factor needs to be reevaluated at the new crack size for the loading in the next cycle. Hence, it becomes necessary to integrate the expression in Eq. (6) through a cycle by cycle procedure. Each cycle involves the computation of ΔK using a finite element analysis represented by Ψ .

$$\Delta K_{eqv} = \Psi(a, L, \theta) \quad (7)$$

Repeated evaluation of the finite element analysis in Eq. (7) renders the aforementioned cycle by cycle integration extremely expensive, perhaps impossible in some cases. Hence, it is necessary to substitute the finite element evaluation by an inexpensive surrogate model. Different kinds of surrogate models (polynomial chaos, support vector regression, relevance vector regression, and Gaussian Process interpolation) have been explored and the Gaussian process modeling technique has been employed in this paper. A few runs of the finite element analysis are used to train this surrogate model and then, this model is used to predict the stress intensity factor for other crack sizes and loading cases (for which finite element analysis has not been carried out).

3.3 Construction of Gaussian Process Surrogate Model

A Gaussian process (GP) response surface approximation is constructed to capture the relationship between the input variables (a, L, θ) and the output variables (ΔK) in Eq. (5), using only a few sample points within the design space. The details of this interpolation technique are available in literature (Rasmussen, 1996; Santner, 2003; McFarland, 2007).

The basic idea of the GP model is that the response values Y (K_{eqv} in this case), are modeled as a group of multivariate normal random variables, with a defined mean and covariance function. The benefits of GP modeling is that the method requires only a small number of sample points (usually 30 or less), and is capable of capturing highly nonlinear relationships that exist between input and output variables without the need for an explicit functional form. Additionally, Gaussian process models can be used to fit virtually any functional form and provide a direct estimate of the uncertainty associated with all predictions in terms of model variance. The framework of Gaussian process modeling is shown in Figure 3.

Suppose that there are n training points, $x_1, x_2, x_3 \dots x_n$ of a d -dimensional input variable (the input variables being the crack size and loading conditions here), yielding the resultant observed random vector $Y(x_1), Y(x_2), Y(x_3) \dots Y(x_n)$. R is the $m \times m$ matrix of correlations among the training points. An exponential correlation function has

been suggested by researchers in the past (Bichon et al., 2008).

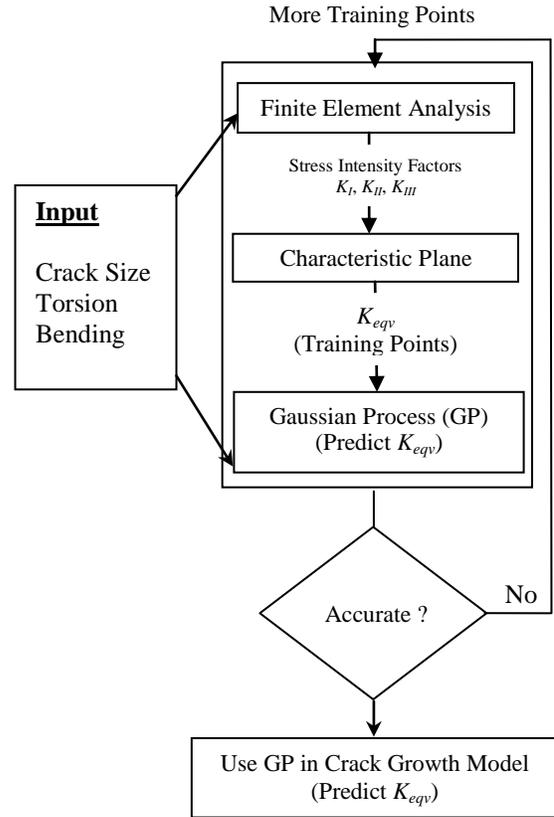


Figure 3. Construction of Surrogate Model

$$Y^* = E(Y|x^*) = f^T(x^*)\beta + r^T(x^*)R^{-1}(Y - F\beta) \quad (8)$$

$$\sigma_{Y^*} = \text{Var}(Y|x^*) = \lambda(1 - r^T R^{-1} r) \quad (9)$$

In Eq. (8) and Eq. (9), F is a matrix with rows of trend functions $f^T(x_i)$, r is the vector of correlations between x^* and each of the training points, β represents the coefficients of the trend function. A constant trend function has been reported to be sufficient (Sacks et al., 1989). McFarland (2007) discusses the implementation of this method in complete detail.

3.4 Crack Propagation Analysis

This section explains the method used to calculate the final crack size as a function of number of load cycles. The procedure involves the evaluation of the integral in Eq. (4). As explained in Section 3.3, this needs to be done cycle by cycle and the Gaussian process surrogate model is used to predict the equivalent stress intensity factor in each cycle. Starting with the equivalent initial flaw size a_0 , the equations (Eq. (1) – Eq. (6)) described in Section 3.1 are

used to calculate the final crack size A after N loading cycles. This entire procedure is summarized in Figure 4.

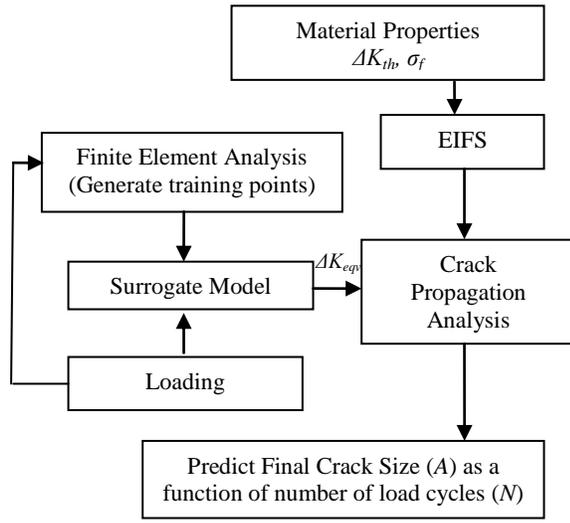


Figure 4. Crack Propagation Analysis

The framework shown in Figure 4 for crack growth prognosis is deterministic and does not account for errors and uncertainty. Uncertainty can be associated with each of the blocks in Figure 4 and accounted for in crack growth prediction. The following section investigates these sources of uncertainty and Section 5 incorporates them into the crack growth analysis methodology.

4. SOURCES OF UNCERTAINTY

This section discusses the various sources of uncertainty and errors that are part of the crack growth framework summarized in Section 3.5 and proposes methods to handle different types of uncertainty. The material properties used to calculate the equivalent initial flaw size are measured using experiments and have variability, causing variability in EIFS. Further, these experimental data may be sparse and the uncertainty in data needs to be accounted for. The crack growth law used for crack propagation is usually estimated through curve fitting of experimental data. To account for model uncertainty, a (normally distributed) error term is added to the crack growth equation and the model coefficients of the crack growth law are treated as random variables. In each cycle of loading, the stress intensity factor is calculated as a function of current crack size, loading and geometry. Repeated finite element analyses are avoided by the use of inexpensive surrogate models and the output of the surrogate model is not accurate. Further, the training points calculated using finite element analyses are prone to solution approximation and discretization errors. Further, the loading itself is considered to be random – a variable amplitude multi-axial loading case is demonstrated in this paper. These various sources of uncertainty can be classified into three different types – physical variability, data uncertainty and model uncertainty - as shown below.

I. Physical Variability

- a. Loading
- b. Equivalent initial flaw size
- c. Material Properties (Fatigue Limit, Threshold Stress Intensity Factor)

II. Data Uncertainty

- a. Material Properties (Fatigue Limit, Threshold Stress Intensity Factor)

III. Model Uncertainty/Errors

- a. Crack growth law uncertainty
- b. Uncertainty in calculation of Stress Intensity factor
- c. Discretization error in finite element analysis
- d. Uncertainty in surrogate model output

(Note: Variations in geometry and boundary conditions are sources of physical variability. These variations are not considered in this research work. However, these can be included in the proposed framework by constructing different finite element models (for different geometry and boundary conditions) and use these runs to train the Gaussian process surrogate model. Hence, these parameters are treated as inputs to the surrogate model and sampled randomly in the uncertainty quantification procedure explained later in Section 5.)

The following subsections discuss each source of uncertainty in detail and propose methods to handle them.

4.1 Physical Variability in Loading Conditions

The loading on practical structures is rarely deterministic and it is difficult to quantify the uncertainty in loading. For the purpose of illustration, variable amplitude multi-axial (bending, tension and torsion) loading is considered in this paper. A loading history consists of a series of blocks of loads, the loading amplitude being constant in each block. In this paper, the block length is assumed to be a random variable and the maximum and minimum amplitudes in each block are also treated as random variables. A sample loading history is shown in Figure 5.

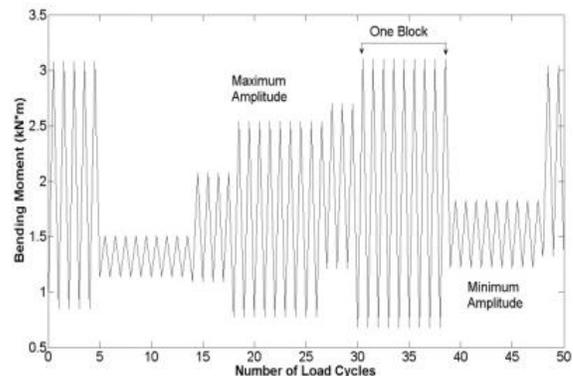


Figure 5. Sample loading history

To generate one block of loading, first a block length is selected and then a maximum amplitude value and a minimum amplitude value is selected for that block. The entire loading history is generated by repeating this process and creating several successive blocks.

4.2 Physical Variability in EIFS

The equivalent initial flaw size derived in Eq. (3) depends on ΔK_{th} , the equivalent mode-I threshold stress intensity factor, $\Delta\sigma_f$, the fatigue limit of the specimen and the geometry factor Y which in turn depends on the geometry of the structural component and the configuration of the crack. This is a deterministic quantity and can be estimated using finite element analysis. The distributions for the material properties, ΔK_{th} and $\Delta\sigma_f$ are characterized using data obtained from experimental testing. This is explained in Section 4.3. Having obtained the statistical distributions of ΔK_{th} and $\Delta\sigma_f$, the distribution of a_0 , the equivalent initial flaw size, can be calculated.

4.3 Data Uncertainty in Material Properties (to characterize distributions ΔK_{th} and $\Delta\sigma_f$)

This section proposes a general methodology to characterize uncertainty in input data, from which statistical distributions need to be inferred. This method is illustrated using experimental data available in literature to characterize the distribution of threshold stress intensity factor (ΔK_{th}) and fatigue limit ($\Delta\sigma_f$). McDonald et al. (McDonald et al., 2009) proposed a method to account for data uncertainty, in which in the quantity of interest can be represented using a probability distribution, whose parameters are in turn represented by probability distributions.

Consider a random variable X whose statistics are to be determined from experimental data, given by $\mathbf{x} = \{x_1, x_2 \dots x_n\}$. For the sake of illustration, suppose that the random variable X follows a normal distribution, then the parameters (\mathbf{P}) of this distribution, i.e. mean and variance of X can be estimated from the entire data set \mathbf{x} . However, due to sparseness of data, these estimates of mean and variance are not accurate. Using resampling techniques such as bootstrapping method (Efron and Tibshirani, 1993), jackknifing (Efron, 1979) etc. the probability distributions ($f_{\mathbf{P}}(\mathbf{P})$) of the parameters (\mathbf{P}) can be calculated. Hence for each instance of a set of parameters (\mathbf{P}), X is defined by a particular normal distribution. However, because the parameters (\mathbf{P}) themselves are stochastic, X is defined by a family of normal distributions. For a detailed implementation of this methodology, refer McDonald et al., 2009.

Note that the aforementioned resampling techniques are useful, when considerable amount of data are available. For example, if 30 data points are needed to construct a meaningful probability distribution, then resampling techniques are not needed. Resampling techniques are very

useful if there are, say 8 to 20 data points. Resampling techniques are less meaningful when there are less than 5 data points.

This paper uses resampling techniques to calculate the distribution of the parameters (\mathbf{P}), however does not define a family of distributions. Instead, it recalculates the distribution of the random variable X , using principles of conditional probability (Haldrar and Mahadevan, 2000). Thus X follows a probability distribution conditioned on the set of parameters (\mathbf{P}). Hence the distribution of X is denoted by $f_{X|\mathbf{P}}(x)$. However, in this case, the parameters are represented by probability distributions $f_{\mathbf{P}}(\mathbf{P})$. Hence, the unconditional probability distribution of X ($f_X(x)$) can be calculated as shown in Eq. (10).

$$f_X(x) = \int f_{X|\mathbf{P}}(x) f_{\mathbf{P}}(\mathbf{P}) d\mathbf{P} \quad (10)$$

The integral in Eq. (10) can be evaluated through quadrature techniques or advanced sampling methods such as Monte Carlo integration or Markov chain Monte Carlo Integration. Hence, the unconditional distribution of X which accounts for uncertainty in input data can be calculated. In this paper, this method has been used to characterize the uncertainty in threshold stress intensity factor (ΔK_{th}) and fatigue limit ($\Delta\sigma_f$).

4.4 Uncertainty in Crack Growth Model

There are more than 20 different crack growth laws (e.g., Paris law, Foreman's equation, Weertman's equation) proposed in literature. The mere presence of many such different models explains that none of these models can be applied universally to all fatigue crack growth problems. Each of these models has its own limitations and uncertainty. In this paper, a modified Paris law has been used for illustration, however, the methodology can be implemented using any kind of crack growth model. The uncertainty in crack growth model can be subdivided into two different types: crack growth model error and uncertainty in model coefficients. If ε_{cg} is used to denote the crack growth model error, then the crack growth law can be expressed as:

$$da/dN = \phi^r C (\Delta K)^n (1 - \Delta K_{th}/\Delta K)^m + \varepsilon_{cg} \quad (11)$$

An estimate of ε_{cg} can be obtained while calibrating the model parameters using statistical data fitting tools. The model coefficients in Paris law are C and n , and the uncertainty in these parameters can be represented through probability distributions. The stress intensity factor ΔK , as explained earlier is calculated using the Gaussian process surrogate model as explained in Section 3. The various sources of uncertainty in this process are addressed in Section 4.5.

4.5 Errors in Stress Intensity Factor Calculation

As explained in Section 3, a Gaussian process model is used to calculate the stress intensity factor ΔK . This is done in two stages. First, a few finite element analysis runs are required to train the GP model. Second, the GP model is used to predict the stress intensity factor as explained in Section 3.3. Each of these two steps has associated errors and uncertainty. Finite element solutions are subject to discretization errors, whereas the prediction of any low-fidelity model such as the GP model also has error. These two issues are discussed in this subsection.

4.5.1 Discretization Error in Finite Element Analysis

Theoretically, an infinitesimally small mesh size will lead to the exact solutions but this is difficult to implement in practice. Hence, finite element analyses are carried at a particular mesh size and the error in the solution, caused due to discretization needs to be quantified. Several methods are available in literature but many of them quantify some surrogate measure of error to facilitate adaptive mesh refinement. The Richardson extrapolation (RE) method has been found to come closest to quantifying the actual discretization error and this method has been extended to stochastic finite element analysis by Rebba (Richards, 1997; Rebba, 2005). It should be noted that the use of Richardson extrapolation to calculate discretization error requires the model solution to be convergent and the domain to be discretized uniformly (uniform meshing) (Rebba et al., 2004). Sometimes, in the case of coarse models, the assumption of monotone truncation error convergence is not valid. In the Richardson extrapolation method, the discretization error due to grid size, for a coarse mesh is given by Eq. (12).

$$\varepsilon_h = (f_1 - f_2) / (r^p - 1) \quad (12)$$

In Eq. (12), f_1 and f_2 are solutions for a coarse mesh and a fine mesh respectively. If the corresponding mesh sizes were denoted by h_1 and h_2 , then the grid refinement ratio, denoted by r is calculated as h_2/h_1 . The order of convergence of p is calculated as:

$$p = \log((f_3 - f_2) / (f_2 - f_1)) / \log(r) \quad (13)$$

In Eq. (13), f_3 represents the solution for a coarse mesh of size h_3 , with the same grid refinement ratio, i.e. $r = h_3/h_2$. The solutions f_1, f_2, f_3 are dependent on the inputs (loading, current crack size, aspect ratio and angle of orientation) to the finite element analysis and hence the error estimates are also functions of these input variables. For each set of inputs, a corresponding error is calculated and this error is added to the (coarse mesh) solution from finite element analysis to calculate the true solution. Hence a true solution

is associated with each set of inputs and these values are used as training points for the surrogate model.

4.5.2 Uncertainty in the Surrogate Model Output

Several finite element runs for some combination of input-output variable values are used to train the Gaussian process surrogate model in this paper. Then, these surrogate models can be used to evaluate the stress intensity factor for other combinations of input variable values. GP models, as explained in Section 3.3, model the output as a sum of Gaussian variables and hence, inherently produce an output which is normally distributed. The expressions for mean and variance of the output of the GP model were given in Eq. (8) and Eq. (9) respectively. The output of the GP (ΔK_{eq}) model is a random normal variable and in each cycle, the value for ΔK_{eq} is sampled from this distribution.

(Note: The GP model is used as a surrogate for the deterministic finite element model and the variance of the GP output accounts only for the uncertainty in replacing the original model with a Gaussian process and does not account for the uncertainty in the inputs to the model. The variance of the output is only dependant on the “form” of the surrogate model. For example, a linear surrogate model will lead to constant variance at untrained locations but unknown distribution type (Seber and Wild, 1989). The advantage in using a Gaussian process surrogate model is that not only the output variance can be calculated but also the distribution type can be proved to be Gaussian (McFarland, 2007).)

The Gaussian process model output, i.e. the stress intensity factor is used in the crack growth equation to predict the crack size as a function of number of cycles as explained earlier in Section 3. The following section incorporates all these sources of uncertainty into the crack growth prediction methodology described in Section 3.

5. UNCERTAINTY IN CRACK GROWTH

Section 3 proposed a methodology that can be used for crack growth of structures with complicated geometry and subjected to multi-axial loading. This procedure was summarized using a step-by-step flowchart in Figure 4. Section 4 investigated the various sources of uncertainty in the crack growth prediction methodology and proposed methods to handle them. A brief summary of the various sources of uncertainty is given below.

I. PHYSICAL VARIABILITY

- Variable amplitude multi-axial loading cycles are generated by considering random block lengths and random amplitudes within each block.
- The equivalent initial flaw size (EIFS) is represented by a probability distribution that accounts for the variability in material parameters, the threshold stress intensity factor and fatigue limit.

- c. The material properties (fatigue limit, threshold stress intensity factor) are represented by probability distributions, inferred from experimental data.

II. DATA UNCERTAINTY

- a. The uncertainty in data used to calculate the statistics of material properties (fatigue limit, threshold stress intensity factor) is addressed by using a sampling based approach that calculates a family of probability distributions for each material parameter. Then, this family of distributions is integrated into one single probability distribution (for each property) using the principles of conditional and total probability.

III. MODEL UNCERTAINTY/ERRORS

- a. The uncertainty in crack growth model is handled by adding an error term to the crack growth law and by representing the model parameters as random variables.
- b. The calculation of stress intensity factor in each cycle of crack growth is facilitated using a Gaussian process surrogate model.
- c. The discretization error in finite element analysis is calculated using Richardson extrapolation and added to the results of FEA before training the surrogate model.

The uncertainty (calculated as the variance) in the surrogate model output is modeled as a Gaussian variable calculation from regression results and hence, the prediction of the surrogate model, i.e. the Stress intensity factor is represented as a normal distribution.

This section presents a sampling based strategy to combine all the different sources of uncertainty and thereby quantify the overall uncertainty in crack growth prediction as a function of number of loading cycles (N). The various steps in this procedure are outlined here.

- I. Generate training points for the Gaussian process surrogate model. This is done through finite element analysis and then by calculating the discretization error in each of the runs. The discretization errors are added to the solutions of finite element analysis and used to train the Gaussian process surrogate model. Hereon, the GP model can be used to calculate the stress intensity factor as a function of crack size, loading, aspect ratio and angle of orientation.
- II. Generate a loading history. First, randomly select a block length and then randomly select a maximum amplitude value and a minimum amplitude value for that particular block. Repeat the process till the number of cycles (N) is reached.
- III. Sample an EIFS value from the statistical distribution calculated in Section 4.1 and Section 4.2.
- IV. Use the deterministic procedure for crack growth analysis to calculate the final crack size at the end of N cycles. However, in each loading cycle, the stress intensity factor calculated from the GP model is a

random normal variable and hence generate a random sample of stress intensity factor in each cycle. Also, the crack growth model error (ε_{cg}) is sampled in every cycle.

In this algorithm, Step I is a deterministic step while Step II, Step III and Step IV are probabilistic. Using this algorithm, the crack size after N cycles can be calculated for a particular load history that was generated in Step II. Using Monte Carlo Sampling, Steps II, III and IV can be repeated again and again, each leading to a final crack size at the end of N cycles. This can be used to characterize the distribution of final crack size at the end of N cycles. By varying N , the distribution of final crack size can be obtained as a function of the number of cycles (N). This information can be used to calculate the reliability of the structural component as a function of number of load cycles. Suppose that the component is supposed to have failed if the crack size is greater than a critical crack size (A_c), then the probability of failure can be calculated as a function of load cycles.

Note that the proposed methodology for uncertainty quantification is applicable to any model-based technique for fatigue crack growth prognosis. Thus any appropriate combination of fatigue crack growth models can be used instead of the models used in this paper for implementing the proposed uncertainty quantification methodology.

6. NUMERICAL EXAMPLE

This section illustrates the proposed methodology to quantify the uncertainty in crack growth analysis through a numerical example.

6.1 Description of the Problem

A two radius hollow cylinder with an elliptical crack in fillet radius region is considered. Coarse Full Model This problem consists of modeling an initial semi-circular surface crack configuration and allowing the crack shape to develop over time into a semi-elliptical surface crack. This is shown in Figure 6.

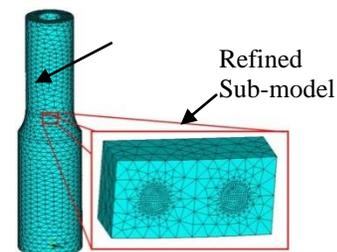


Figure 6. Surface Crack in a Cylindrical Structure

The finite element software package ANSYS (ANSYS, 2007) version 11.0 is used to build and analyze the finite element model. The crack configuration is built by extruding a projection of the semi-circular crack through the

most body at the crack location. The immediate volumes on either side of the crack face are identified and subdivided in order to allow for SIF evaluation at various locations along the crack front. The crack faces (coinciding upper and lower surfaces of the previously mentioned volumes) are then modeled as surface to surface contact elements (CONTACT174 and TARGET170 elements) in order to prevent the surface penetration of the crack's upper and lower surfaces. The augmented Lagrangian method is the algorithm used for contact simulation. Additionally, friction effect is included in the material properties of the contact element, in which a Coulomb friction model is used. This model defines an equivalent shear stress which is proportional to the contact pressure and the friction coefficient. Friction coefficients between two crack faces are difficult to measure and are generally assumed to vary between 0 and 0.5 (Liu et al., 2007). The friction coefficient, μ is assumed to be equal to 0.1.

Since the primary quantity of interest is the stress intensity factor at the crack tip, the volume along the crack front is subdivided into many smaller blocks, which allows for better mesh control and enables SIF evaluation at various locations along the crack front. The crack region is constructed within a submodel of the uncracked body. The submodel technique is based on the St. Venant's principle, which states that if an actual distribution of forces is replaced by a statically equivalent system, the distribution of stress and strain is altered only near the regions of load application. The sub-modeling technique facilitates accurate stress intensity factor solutions all along the crack front which can be used for crack growth analysis.

Table 1 and Table 2 list the material and geometrical properties of the specimen under study.

Aluminium 7075- T6	
Modulus of Elasticity	72 GPa
Poisson Ratio	0.32
Yield Stress	450 MPa
Ultimate Stress	510 MPa

Table 1 Material properties

Cylinder Properties	
Length	152.4 mm
Inside Radius	8.76 mm
Outside Radius (Narrow Sect)	14.43 mm
Outside Radius (Wide Sect)	17.78 mm

Table 2 Geometrical Properties

In reality, these parameters in Table 2 and Table 3 may be variable and might require probabilistic treatment. However,

as mentioned earlier, physical variability in the geometry of the structure, Young's modulus, Poisson ratio, boundary conditions, friction coefficient between crack faces, etc are treated to be deterministic in this paper. The following subsection discusses the numerical implementation of the uncertainty quantification procedure.

6.2 Algorithm for Uncertainty Quantification

The numerical details of the different sources of uncertainty are presented in this section. They are given step-wise in the same order as in Section 5.

- I. Finite element analyses are run for 10 different crack sizes, 6 different loading cases, two angles of orientation and three different aspect ratios, amounting to 360 training points to construct the surrogate model. For each solution, three different meshes are considered and the discretization error is quantified as explained in Section 4.4.1. The discretization error is added to the finite element analysis solution at each training point and the Gaussian process model is trained to predict the stress intensity factor.
- II. Multi-axial variable amplitude loading cycles are generated by considering blocks of equal amplitude within one entire loading history. The block length is assumed to be a uniform distribution (U(0,500)) and the maximum amplitude and minimum amplitude for that block are assumed to follow normal distributions ($N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$ where $\mu_1, \sigma_1, \mu_2, \sigma_2$ are uniformly distributed on the intervals [20, 28], [2, 6], [8, 16], and [2, 6] respectively, in KNm).
- III. The distribution of EIFS is characterized using the data used by Liu and Mahadevan (Liu and Mahadevan, 2008). However, the current research work accounts for uncertainty in data and treats the parameters of threshold stress intensity factor and fatigue limit as random variables as well. The distribution (conditioned on its parameters) of EIFS is assumed to be lognormal (with parameters λ, ζ), with the λ following a normal distribution (mean = -7.60 and standard deviation = 0.50) and ζ following a lognormal distribution (mean = 0.22 and standard deviation = 0.10). The unconditional distribution of EIFS is calculated using the integral in Eq. (8). Samples of EIFS are drawn from this distribution.
- IV. Paris law is used for crack growth propagation. The model parameter C (mean = 6.5 E-13 and standard deviation = 4E-13) is chosen to be lognormally distributed whereas m ($m = 3.9$) is treated as a deterministic quantity. These are identical to the distributions used by Liu and Mahadevan (Liu and Mahadevan, 2009). In each loading cycle, the values of stress intensity factor and crack growth model error (ε_{cg}) are sampled from probability distributions. While the stress intensity factor (calculated using the Gaussian process surrogate model) is a Gaussian variable (as

explained in section 4.5.2), a 5% Gaussian white noise is used to represent the crack growth model error. The latter quantity is chosen to be normal (Seber and Wild, 1989) because it represents a fitting error while calculating the coefficients of modified Paris' law. Note that the true value of this noise can be estimated from actual experimental data, and a 5% value is chosen only for illustration.

Using the sampling-based framework in Section 5, the probability distribution of the final crack size is calculated as a function of the total number of cycles. A Monte Carlo simulation using 5000 runs is used to calculate the probability distribution of crack size as a function of number of load cycles. The mean, median and 90% prediction bounds of the final crack size are shown in Figure 7.

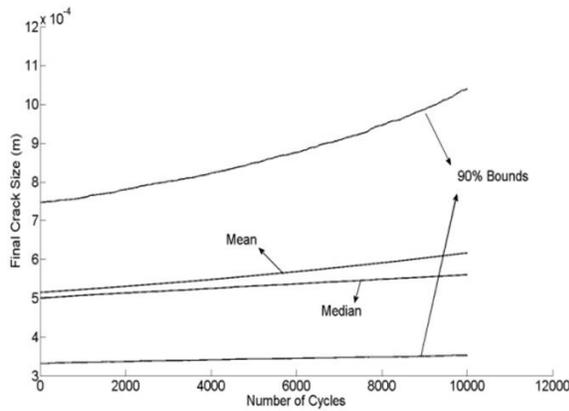


Figure 7. Mean, Median and 90% Bounds

In Figure 7, the growth of the crack is shown as a function of number of load cycles. As the number of cycles increase, there is more uncertainty and hence, the 90% prediction bounds are wider. This is due to the fact that each additional loading cycle imparts more randomness arising from variability in loading, variability in crack size at the end of previous cycle, uncertainty in the prediction of stress intensity factor, etc. To illustrate the increase in uncertainty, the standard deviation of crack size is calculated as a function of number of load cycles and plotted in Figure 8.

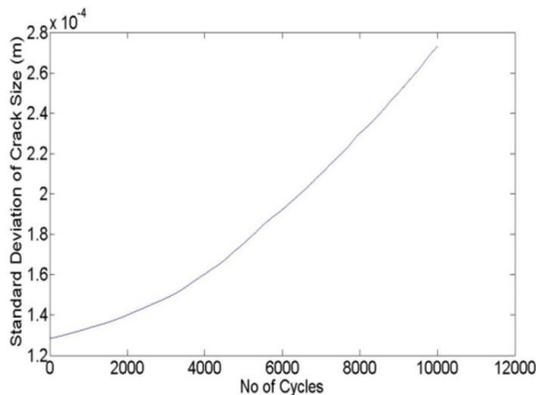


Figure 8. Standard Deviation of Final Crack Size

Figure 8 clearly shows the increase in uncertainty with number of load cycles. While the standard deviation of the initial crack size is low, it increases by about 500% at the end of 5000 load cycles. This increase is due to accumulation of different sources of uncertainty in each loading cycle, i.e. loading uncertainty, surrogate modeling errors and crack growth model errors.

Finally, the reliability of the structural component is also evaluated. A critical crack size of 2.54 mm (approximately 0.1 inch) is assumed for the purpose of illustration and the probability of failure is estimated as a function of number of load cycles and plotted in Figure 9. From Figure 9, it is seen that the probability of failure is negligible for about 3500 load cycles and it gradually increases after 4000 cycles.

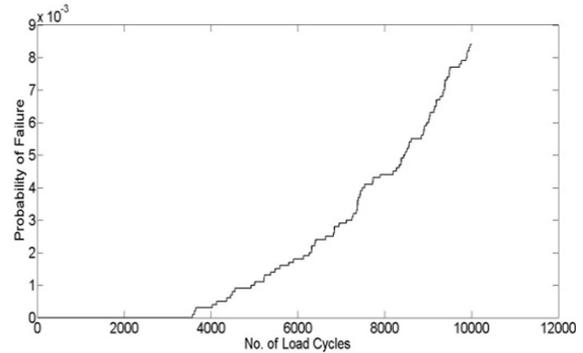


Figure 9. Probability of Failure vs. No. Load Cycles.

There are two reasons for the observed increase in increase of failure probability. Firstly, the crack is growing in size and secondly, the uncertainty in the estimated crack size also increases with each loading cycle. After 10000 cycles of loading, the probability of failure is approximately equal to 0.01.

6.3 Individual Contributions of Uncertainty

The previous subsection presented the effect of all the different sources of uncertainty in the final distribution of crack size. The current subsection calculates the marginal contributions of each source of uncertainty in the overall results of crack growth calculation. Such an analysis would identify which sources of uncertainty are critical and what the analyst must do in order to reduce the overall uncertainty in crack growth prediction.

To calculate the contribution of one particular kind of uncertainty, all other quantities are assumed to be deterministic (at their mean values) and the results of this analysis are compared with the results of Section 6.2, where all sources of uncertainty were accounted. The individual contributions of each uncertainty are tabulated in Table 3. This approach facilitates resource allocation trade-offs between model refinement and data collection for the purpose of reduction in overall uncertainty. If the contribution from the uncertainty in a particular model is

high, then it is preferable to refine the model in order to reduce the uncertainty in the crack growth prediction. If the contribution from the data is high (as seen in Table 3), then it is preferable to collect more data to reduce the overall uncertainty in the crack growth prediction. Note that experimental data on material properties is used to characterize the distribution of the equivalent initial flaw size; hence, a reduction in data uncertainty would lead to reduction in EIFS uncertainty, and thus the overall uncertainty in the crack growth prediction.

Sources of Uncertainty Considered	Final Crack Size		
	Mean (mm)	Std (mm)	COV
All	0.617	0.273	0.4424
Loading	0.592	0.068	0.1152
Crack Growth Model	0.544	0.023	0.0421
Data Uncertainty	0.547	0.151	0.2767
EIFS Uncertainty	0.544	0.134	0.2463
GP Model Uncertainty	0.544	5.33E-5	9.81E-6

Table 3. Individual Contributions of Uncertainty

The above decision inferences are qualitative; future work needs to quantify the contributions of various sources of uncertainty to the overall uncertainty in the model prediction. Quantitative sensitivity analysis techniques may be pursued for this purpose.

7. SUMMARY

This paper investigated the various sources of uncertainty in a fatigue crack growth prognosis problem and illustrated the proposed methods to quantify the overall uncertainty in crack growth prediction for structures with complicated geometry and multi-axial variable amplitude loading. The concept of equivalent initial flaw size was used to replace small crack growth analysis and use a long crack growth model, specifically modified Paris law, for crack propagation. A characteristic plane approach was used to calculate an equivalent stress intensity factor in the presence of multi-axial loading conditions. Crack growth under variable amplitude loading is modeled by including retardation effects in the modified Paris law. Expensive finite element analysis was replaced by an inexpensive surrogate, i.e. the Gaussian process model, to evaluate the stress intensity factor in each cycle for use in crack growth law. Several sources of uncertainty – physical variability, data uncertainty and modeling errors - were included in the crack growth analysis procedure. Physical variability included loading conditions and material properties such as threshold stress intensity factor and fatigue limit. The uncertainty in data used to characterize these parameters was accounted for. Three different types of modeling errors – discretization errors, surrogate modeling error and crack

growth model error – were considered in this paper. A probabilistic methodology was proposed to incorporate these sources of uncertainty into the crack growth prediction methodology. A Monte Carlo based sampling approach is used to calculate the distribution of crack size as a function of number of loading cycles. By defining a suitable serviceability criterion (for example, crack size being greater than a critical value), the reliability of the structural component is calculated as a function of number of loading cycles. The methods developed for uncertainty quantification are applicable to any model-based fatigue crack growth prognosis procedure, and are not confined to the illustrative crack growth models used in this paper.

This research work also reported the individual contributions of various sources of uncertainty to the overall uncertainty in crack growth prediction. This kind of study is popularly called as global sensitivity analysis and the method presented in this paper is a heuristic approach only. Rigorous methods for sensitivity analysis have been developed by several researchers around the world and future work would involve the application of these methods to crack growth analysis problems. This study would analyze the significance of the individual contributors, assess and rank their importance, and quantify the benefits from reducing the uncertainty in the important contributors. Further, this research work considered the growth of coplanar cracks only. Non-coplanar cracks will be modeled and included in uncertainty analysis in future. Also, additional sources of variability, uncertainty, and error, such as variability in the coefficient of friction between crack surface, material properties such as Young's modulus, geometrical properties, model uncertainty in the treatment of mixed-mode cracking, and variance introduced by introduced by Monte Carlo sampling will be considered future.

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NOMENCLATURE

A	Crack size
N	Number of loading cycles
θ	Equivalent initial flaw size
ΔK_{th}	Threshold stress intensity factor
ΔK	Stress intensity factor in each cycle

σ_f	Fatigue limit
Y	Geometry factor
ϕ^r	Wheeler's retardation coefficient
C, m, n	Parameters of modified Paris' law

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